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PERFECTLY SUPPORTABLE SEMIGROUPS ARE σ-DISCRETE IN EACH HAUSDORFF SHIFT-INVARIANT TOPOLOGY

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Abstract. In this paper we introduce perfectly supportable semigroups and prove that they are σ-discrete in each Hausdorff shift-invariant topology. The class of perfectly supportable semigroups includes each semigroup S such that $\mathsf{FSym}(X) \subset S \subset \mathsf{FRel}(X)$ where $\mathsf{FRel}(X)$ is the semigroup of finitely supported relations on an infinite set X and $\mathsf{FSym}(X)$ is the group of finitely supported permutations of X.

1. Introduction

The problem considered in this paper traces its history back to S.Ulam who asked in [5, p.178] and [6] if for some infinite set X the group $\mathsf{Sym}(X)$ of bijections of X carries a non-discrete locally compact group topology. The Ulam's problem was solved in negative in 1967 by E.Gaughan [3]. This result motivated the following problem posed in [4]:

Problem 1.1. *Let X be a set of cardinality $|X| = \mathfrak{c}$ and let $\mathsf{FSym}(X)$ be the group of bijections $f : X \to X$ having finite support $\mathsf{supt}(f) = \{x \in X : f(x) \neq x\}$. Does $\mathsf{FSym}(X)$ admit a non-discrete locally compact Hausdorff group topology?*

In [4] it was shown that for any infinite set X the group $\mathsf{FSym}(X)$ does not admit a compact Hausdorff group topology. A negative answer to Problem 1.1 was given in [1], where the following theorem was proved:

Theorem 1.2 (Banakh-Guran-Protasov)**.** *For any set X the group $\mathsf{FSym}(X)$ is σ-discrete in each Hausdorff shift-invariant topology on $\mathsf{FSym}(X)$. Consequently, each Baire Hausdorff shift-invariant topology on $\mathsf{FSym}(X)$ is discrete.*

In this paper we generalize Theorem 1.2 to the class of perfectly supportable semigroups. Such semigroups are defined in Section 2. Our main result is Theorem 3.2, proved in Section 3. It implies Corollary 3.5 saying that each perfectly supportable semigroup is σ-discrete in each Hausdorff shift-invariant topology. In Section 4 we present an example of perfectly supportable semigroup $\mathsf{FRel}(X)$, which contains many other perfectly supportable (semi)groups as sub(semi)groups.

2. supt-Semigroups and supt-perfect semigroups

In this section we define the classes of semigroups called supt-semigroups and supt-perfect semigroups.

Definition 2.1. A supt-*semigroup* is a pair (S, supt) consisting of a semigroup S and a function $\mathsf{supt} : S \to 2^X$ with values in the power-set 2^X of some set X, such that for each elements $f, g \in S$ we get:

(1) $\mathsf{supt}(fg) \subset \mathsf{supt}(f) \cup \mathsf{supt}(g)$,
(2) $fg = gf$ if $\mathsf{supt}(f) \cap \mathsf{supt}(g) = \emptyset$.

The function $\mathsf{supt} : S \to 2^X$ is called the *support map* of the supt-semigroup (S, supt). The set $\mathsf{supt}(S) = \bigcup_{a \in S} \mathsf{supt}(a) \subset X$ is called *the support of S*.

A typical example of a supt-semigroup is the group $\mathsf{Sym}(X)$ of all bijections $f : X \to X$ of a set X, endowed with the support map $\mathsf{supt} : f \mapsto \{x \in X : f(x) \neq x\}$. In Section 4 we shall describe another supt-semigroup $\mathsf{Rel}(X)$, which contains $\mathsf{Sym}(X)$ (and many other semigroups) as a supt-subsemigroup.

Definition 2.1 implies the following proposition-definition.

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Proposition 2.2. *Let (S, supt) be a supt-semigroup with support $X = \mathsf{supt}(S)$.*

(1) *For each infinite cardinal λ the set $S_{<\lambda} = \{f \in S : |\mathsf{supt}(f)| < \lambda\}$ is a subsemigroup of S;*
(2) *For each subset $F \subset X$ the set $S(F) = \{f \in S : \mathsf{supt}(f) \subset F\}$ is a subsemigroup of S;*
(3) *For each subsemigroup $T \subset S$ the pair $(T, \mathsf{supt}|T)$ is a supt-semigroup.*

Definition 2.3. A supt-semigroup (S, supt) with support set $X = \mathsf{supt}(S)$ is called a supt-*finitary semigroup* if

(1) each element $f \in S$ has finite support $\mathsf{supt}(f)$ and
(2) for each finite subset $F \subset X$ the subsemigroup $S(F) = \{f \in S : \mathsf{supt}(f) \subset F\}$ of S is finite.

This definition implies that each supt-finitary semigroup S coincides with its subsemigroup $S_{<\omega}$ consisting of finitely supported elements of S.

A typical example of a supt-finitary supt-semigroup is the group $\mathsf{FSym}(X) = \mathsf{Sym}(X)_{<\omega}$ of finitely supported bijections of a set X.

To define supt-perfect semigroups, we need to recall some information on centralizers.

For a semigroup S, an element $f \in S$, and a subset $T \subset S$ by

$$\mathsf{C}(f) = \{g \in G : fg = gf\} \quad \text{and} \quad \mathsf{C}(T) = \bigcap_{f \in T} \mathsf{C}(f)$$

we denote the *centralizers* of f and T, respectively.

The condition (2) of Definition 2.1 implies that $S(F) \subset \mathsf{C}(S(X \setminus F))$ for any finite subset $F \subset X$. For supt-perfect semigroups the converse implication is also true.

Definition 2.4. A supt-semigroup (S, supt) is called a supt-*perfect semigroup* if it is supt-finitary and for each finite subset $F \subset X$ we get $S(F) = \mathsf{C}(S(X \setminus F))$.

Definition 2.5. A semigroup S is called *perfectly supportable* if for some function $\mathsf{supt} : S \to 2^X$ the pair (S, supt) is a supt-perfect semigroup.

Let us discuss the relation of perfectly supportable semigroups with some other classes of semigroups.

Definition 2.6. We shall say that a semigroup S

- is *locally finite* if each finite subset $F \subset S$ lies in a finite subsemigroup $T \subset S$;
- has *finite double centralizers* if for any finite subset $F \subset S$ its double centralizer $\mathsf{C}(\mathsf{C}(F))$ is finite.

Since the double centralizer $\mathsf{C}(\mathsf{C}(F))$ is a subsemigroup that contains F, each semigroup with finite double centralizers is locally finite.

Proposition 2.7. *Each perfectly supportable semigroup has finite double centralizers and hence is locally finite.*

Proof. Let $\mathsf{supt} : S \to 2^X$ be a support map that turns S into a supt-perfect semigroup. We claim that for each finite subset $T \subset S$ its double centralizer $\mathsf{C}(\mathsf{C}(T))$ lies in the finite subsemigroup $S(F)$ where $F = \bigcup_{f \in T} \mathsf{supt}(f)$. Assuming that $\mathsf{C}(\mathsf{C}(T)) \not\subset S(F)$ and taking into account that $S(F) = \mathsf{C}(S(X \setminus F))$, we can find an element $f \in \mathsf{C}(\mathsf{C}(T)) \setminus \mathsf{C}(S(X \setminus F))$. Then for some $g \in S(X \setminus F)$ we get $fg \neq gf$. On the other hand, the condition (2) of Definition 2.1 guarantees that $g \in S(X \setminus F) \subset \mathsf{C}(S(F)) \subset \mathsf{C}(T)$ and hence $f \notin \mathsf{C}(\mathsf{C}(T))$, which is a desired contradiction. □

3. The topological structure of supt-perfect and perfectly supportable semigroups

In this section we shall study the topological structure of supt-perfect and perfectly supportable semigroups endowed with shift-invariant topologies.

A topology τ on a semigroup S is called *shift-invariant* if for every $a \in A$ the left and right shifts

$$l_a : S \to \quad l_a : x \mapsto ax \quad \text{and} \quad r_a : S \to S, \;\; r_a : x \mapsto xa,$$

are continuous. This is equivalent to saying that the semigroup operation $S \times S \to S$ is separately continuous.

Now on each semigroup S we define a shift-invariant T_1-topology (called the semi-Zariski topology) which is weaker than each Hausdorff shift-invariant topology on S.

The *semi-Zariski topology* $\mathfrak{Z}_s$ on a semigroup S is the topology generated by the sub-base consisting of the sets

$$\{x \in S : axb \neq c\} \quad \text{and} \quad \{x \in S : axb \neq cxd\}$$

where $a, b, c, d \in S^1$ and $S^1 = S \cup \{1\}$ stands for the semigroup S with attached external unit $1 \notin S$ (i.e., an element 1 such that $1x = x1 = x$ for all $x \in S^1$). The semi-Zariski topology $\mathfrak{Z}_s$ is a particular case of algebraically determined topologies on G-acts, considered in [2].

The definition of the semi-Zariski topology implies the following simple (but important) fact.

Proposition 3.1. *The semi-Zariski topology $\mathfrak{Z}_s$ on a semigroup S is weaker than each shift-invariant Hausdorff topology on S.*

Now, we shall study the semi-Zariski topology on supt-perfect semigroups. The following theorem is the main result of this paper.

Theorem 3.2. *Let (S, supt) be a supt-perfect semigroup endowed with the semi-Zariski topology $\mathfrak{Z}_s$. For every $n \geq 0$*

1. *the set $S_{\leq n} = \{f \in S : |\mathsf{supt}(f)| \leq n\}$ is closed in S;*
2. *for every $x \in X$ the set $\{f \in S_{\leq n} : x \in \mathsf{supt}(f)\}$ is open in $S_{\leq n}$;*
3. *the set $S_{=n} = \{f \in S : |\mathsf{supt}(f)| = n\}$ is discrete in S;*

Proof. First, we prove two lemmas.

Lemma 3.3. *For each $f \in S$, each point $x \in \mathsf{supt}(f)$, and each finite subset $F \subset X \setminus \{x\}$ either $S(\{x\}) \setminus \mathsf{C}(f) \neq \emptyset$ or else there is an infinite subset $\mathcal{F} \subset S(X \setminus F) \setminus \mathsf{C}(f)$ such that $\mathsf{supt}(g) \cap \mathsf{supt}(h) \subset \{x\} \neq \mathsf{supt}(g)$ for any distinct elements $g, h \in \mathcal{F}$.*

Proof. Assume that $S(\{x\}) \setminus \mathsf{C}(f) = \emptyset$. By induction we shall construct a sequence of elements $(g_i)_{i\in\omega}$ of the semigroup S and an increasing sequence $(F_i)_{i\in\omega}$ of finite subsets of X such that $F_0 = F$ and for every $i \in \omega$ the following conditions hold:

1. $x \notin F_i$,
2. $fg_i \neq g_i f$,
3. $\mathsf{supt}(g_i) \cap F_i = \emptyset$ and $\mathsf{supt}(g_i) \not\subset \{x\}$,
4. $F_{i+1} = F_i \cup \mathsf{supt}(g_i) \setminus \{x\}$.

Assume that for some $i \geq 0$ the set F_i has been constructed. Since $x \in \mathsf{supt}(f)$ and $x \notin F_i$, we get $f \notin S(F_i) = \mathsf{C}(S(X \setminus F_i))$ and hence there is an element $g_i \in S(X \setminus F_i)$ such that $fg_i \neq g_i f$. Put $F_{i+1} = F_i \cup \mathsf{supt}(g_i) \setminus \{x\}$ and observe that $S(\{x\}) \setminus \mathsf{C}(f) = \emptyset$ implies $\mathsf{supt}(g_i) \not\subset \{x\}$. This completes the inductive construction.

It follows that for any number $i < j$ we get

$$\mathsf{supt}(g_j) \setminus \{x\} \subset X \setminus F_j \subset X \setminus F_{i+1} \subset X \setminus \big(\mathsf{supt}(g_i) \setminus \{x\}\big),$$

which implies that the non-empty sets $\mathsf{supt}(g_i) \setminus \{x\}$ and $\mathsf{supt}(g_j) \setminus \{x\}$ are disjoint. Then $g_i \neq g_j$ and $\mathsf{supt}(g_i) \cap \mathsf{supt}(g_j) \subset \{x\}$.

So, $\mathcal{F} = \{g_i\}_{i\in\omega}$ is a required infinite set in $S \setminus \mathsf{C}(f)$ such that $\mathsf{supt}(g) \cap \mathsf{supt}(h) \subset \{x\} \neq \mathsf{supt}(g)$ for any distinct elements $g, h \in \mathcal{F}$. □

Lemma 3.4. *For every $n \in \omega$, each point $f \in S$ has a neighborhood $O_f \in \mathfrak{Z}_s$ in the semi-Zariski topology such that for each $g \in O_f \cap S_{\leq n}$ we get $\mathsf{supt}(f) \subset \mathsf{supt}(g)$.*

Proof. Let $m = |\mathsf{supt}(f)|$ and $\mathsf{supt}(f) = \{x_1, \dots, x_m\}$ be an enumeration of the finite set $\mathsf{supt}(f)$. For every $i \leq m$ by induction we shall construct an increasing sequence of finite sets $F_0 \subset F_1 \subset \dots \subset F_m$ in X and a sequence $\mathcal{F}_1, \dots, \mathcal{F}_m$ of non-empty finite subsets of $S \setminus \mathsf{C}(f)$ such that for every positive $i \leq m$ the following conditions are satisfied:

1. either $\mathcal{F}_i \subset S(\{x_i\})$ or $|\mathcal{F}_i| = n + 1$;
2. $\mathsf{supt}(g) \cap F_{i-1} \subset \{x_i\}$ for each $g \in \mathcal{F}_i$;
3. $\mathsf{supt}(g) \cap \mathsf{supt}(h) \subset \{x_i\}$ for any distinct elements $g, h \in \mathcal{F}_i$;
4. $F_i = F_{i-1} \cup \bigcup_{g\in\mathcal{F}_i} \mathsf{supt}(g)$.

We start the inductive construction letting $F_0 = \mathsf{supt}(f)$. Assume that for some $i < m$ the finite set F_{i-1} has been constructed. Let $F = F_{i-1} \setminus \{x_i\}$ and apply Lemma 3.3 to find a non-empty family $\mathcal{F}_i \subset S(X\backslash F) \setminus \mathsf{C}(f)$ which satisfies the conditions (1) and (3) of the inductive construction. It follows from $\mathcal{F}_i \subset S(X\backslash F)$ that the condition (2) is satisfies. Finally, define the finite set F_i by the condition (4). This completes the inductive construction.

The family $\mathcal{F} = \bigcup_{i=0}^{m} \mathcal{F}_i \subset S \setminus \mathsf{C}(f)$ determines an open neighborhood

$$O_f = \{h \in S : \forall g \in \mathcal{F} \;\; hg \neq gh\}$$

of f in the semi-Zariski topology $\mathfrak{Z}_s$. We claim that $\{x_1, \ldots, x_m\} = \mathsf{supt}(f) \subset \mathsf{supt}(h)$ for each element $h \in O_f \cap S_{\leq n}$. Assume conversely that $x_i \notin \mathsf{supt}(h)$ for some $i \leq m$ and consider two cases.

(i) If $\mathcal{F}_i \subset S(\{x_i\})$, then for each element $g \in \mathcal{F}_i$ we get $\mathsf{supt}(g) \cap \mathsf{supt}(h) \subset \{x_i\} \cap \mathsf{supt}(h) = \emptyset$ and hence $gh = hg$ by the condition (2) of Definition 1.

(ii) If $\mathcal{F}_i \not\subset S(\{x_i\})$, then $|\mathcal{F}_i| = n + 1$ and by the conditions (3) of the inductive construction, the family $\big\{\mathsf{supt}(g) \setminus \{x_i\}\big\}_{g \in \mathcal{F}_i}$ is disjoint and consists of non-empty sets. Since $|\mathcal{F}_i| = n + 1 > |\mathsf{supt}(h)|$, there is an element $g \in \mathcal{F}_i$ such that $\mathsf{supt}(g) \setminus \{x_i\}$ is disjoint with $\mathsf{supt}(h)$. Since $x_i \notin \mathsf{supt}(h)$, the supports $\mathsf{supt}(g)$ and $\mathsf{supt}(h)$ are disjoint and hence $gh = hg$ by the condition (2) of Definition 2.1.

In both cases, we get an element $g \in \mathcal{F}$ with $gh = hg$, which contradicts the choice of $h \in O_f$. $\square$

Now we can finish the proof Theorem 3.2. Let $n \in \omega$.

1. To show that the set $S_{\leq n} = \{f \in S : |\mathsf{supt}(f)| \leq n\}$ is closed in S, fix any element $f \in S \setminus S_{\leq n}$. By Lemma 3.4, the element f has a neighborood $O_f \subset S$ in the semi-Zariski topology $\mathfrak{Z}_s$ such that each $g \in O_f \cap S_{\leq n}$ has support $\mathsf{supt}(g) \supset \mathsf{supt}(f)$, which implies that $n \geq |\mathsf{supt}(g)| \geq |\mathsf{supt}(f)| > n$ and $O_f \cap S_{\leq n} = \emptyset$. So, $S_{\leq n}$ is closed in S.

2. The second item of Theorem 3.2 follows directly from Lemma 3.4.

3. Finally we show that the set $S_{=n} = \{f \in S : |\mathsf{supt}(f)| = n\}$ is discrete in $(S, \mathfrak{Z}_s)$. Fix any element $f \in S_{=n}$ and using Lemma 3.4, find a neighborhood $O_f \in \mathfrak{Z}_s$ of f such that $\mathsf{supt}(f) \subset \mathsf{supt}(h)$ for each $h \in O_f \cap S_{=n}$. Since $|\mathsf{supt}(f)| = n = |\mathsf{supt}(h)|$, we conclude that $\mathsf{supt}(f) = \mathsf{supt}(g)$ and hence $O_f \cap S_{=n}$ lies in the semigroup $S(\mathsf{supt}(f))$, which is finite by the condition (2) of Definition 2.4. Since the semi-Zariski topology $\mathfrak{Z}_s$ satisfies the separation axiom T_1, the open finite subspace $O_f \cap S_{=n}$ of $S_=n$ is discrete and hence f is an isolated point of $S_{=n}$, which means that the space $S_{=n}$ is discrete. $\square$

Let us recall that topological space X is *σ-discrete* if it can be written as a countable union of discrete subspaces. A topology τ on a set X is called *Baire* if for any sequence $(U_n)_{n\in\omega}$ of open dense subsets of the topological space (X, τ) the intersection $\bigcap_{n\in\omega} U_n$ is dense in X. It is well-known that each σ-discrete Baire T_1-space has an isolated point.

Theorem 3.2 implies the main corollary of this paper:

Corollary 3.5. *Each perfectly supportable semigroup S is σ-discrete in its semi-Zariski topology and hence is σ-discrete in each shift-invariant Hausdorff topology on S.*

Another corollary concerns perfectly supportable groups. A group G is called *perfectly supportable* if it is perfectly supportable as a semigroup.

Corollary 3.6. *Each perfectly supportable group G is discrete in each Baire shift-invariant Hausdorff topology on G.*

Proof. Let τ be a Baire Hausdorff shift-invariant topology on a perfectly supportable group G. Since G is a group, the topological space (G, τ) is topologically invariant.

By Corollary 3.5, the topological space (G, τ) is σ-discrete and being Baire, has an isolated point. The topological homogeneity of (G, τ) guarantees that each point of G is isolated and hence the space (G, τ) is discrete. $\square$

4. An example of a supt-perfect semigroup

In this section we consider an important example of a supt-semigroup (which contains many other supt-semigroups as supt-subsemigroups).

Given a set X, consider the semigroup $\mathsf{Rel}(X) = 2^{X\times X}$ of all relations $f \subset X \times X$, endowed with the operation

$$f \circ g = \{(x,z) \in X \times X : \exists y \in X \ \ (x,y) \in f,\ (y,z) \in g\}$$

of composition of relations. For a relation $f \subset X \times X$ by f^{-1} we denote the inverse relation

$$f^{-1} = \{(y,x) : (x,y) \in f\}.$$

Each relation $f \subset X \times X$ can be considered as a multi-valued function assigning to each point $x \in X$ the subset $f(x) = \{y \in X : (x,y) \in f\}$ and to each subset $A \subset X$ the subset $f(A) = \bigcup_{a\in A} f(a)$ of X. Observe that two relations $f, g \subset X \times X$ are equal if and only if $f(x) = g(x)$ for all $x \in X$.

Each function $f : X \to X$ can be identified with the relation $\{(x, f(x)) : x \in X\}$. So, the semigroup $\mathsf{End}(X)$ of all self-maps of X is a subsemigroup of the semigroup $\mathsf{Rel}(X)$. By the same reason, the symmetric group $\mathsf{Sym}(X)$ is a subgroup of the semigroup $\mathsf{Rel}(X)$.

For a relation $f \subset X \times X$ its support is defined as the subset

$$\mathsf{supt}(f) = \big\{x \in X : f(x) \neq \{x\} \text{ or } f^{-1}(x) \neq \{x\}\big\} \subset X.$$

Proposition 4.1. *The pair* $(\mathsf{Rel}(X), \mathsf{supt})$ *is a* supt*-semigroup.*

Proof. Fix any relations $f, g \subset X \times X$.

To show that $\mathsf{supt}(f \circ g) \subset \mathsf{supt}(f) \cup \mathsf{supt}(g)$, take any point $x \in X$ that does not belong to the union $\mathsf{supt}(f) \cup \mathsf{supt}(g)$. Then

$$f \circ g(x) = f(\{x\}) = \{x\}$$

and

$$(f \circ g)^{-1}(x) = g^{-1} \circ f^{-1}(x) = g^{-1}(\{x\}) = \{x\},$$

which means that $x \notin \mathsf{supt}(f \circ g)$.

Next, assuming that $\mathsf{supt}(f) \cap \mathsf{supt}(g) = \emptyset$, we shall show that $f \circ g(x) = g \circ f(x)$ for each point $x \in X$. The inclusion $f \circ g(x) \subset g \circ f(x)$ will follow as soon as we prove that each point $y \in f \circ g(x)$ belongs to the set $g \circ f(x)$. Find a point $z \in g(x)$ such that $y \in f(z)$.

If $y \neq z$, then $f(z) \neq \{z\}$ and $f^{-1}(y) \neq \{y\}$, which implies $y, z \in \mathsf{supt}(f) \subset X \setminus \mathsf{supt}(g)$. Then $x \in g^{-1}(z) = \{z\}$ and hence $x = z$ and

$$y \in \{y\} = g(y) \subset g \circ f(z) = g \circ f(x).$$

Now assume that $y = z$. If $z = x$, then $y = z \in g(x) = g(y) \subset g \circ f(x)$. If $z \neq x$, then $z \in g(x) \neq \{x\}$ and $x \in g^{-1}(z) \neq \{z\}$ and hence $x, z \in \mathsf{supt}(g) \subset X \setminus \mathsf{supt}(f)$. Then $y = z \in g(x) = g(\{x\}) = g \circ f(x)$.

This completes the proof of the inclusion $y \in g \circ f(x)$, which implies that $f \circ g(x) \subset g \circ f(x)$. By analogy we can prove that $g \circ f(x) \subset f \circ g(x)$. □

In the supt-semigroup $\mathsf{Rel}(X)$ consider the supt-subsemigroup

$$\mathsf{FRel}(X) = \{f \in \mathsf{Rel}(X) : \mathsf{supt}(f) \text{ is finite}\}.$$

Observe that for each (finite) subset $F \subset X$ the subsemigroup $\{f \in \mathsf{Rel}(X) : \mathsf{supt}(f) \subset F\}$ can be identified with the (finite) semigroup $\mathsf{Rel}(F)$, which has cardinality $2^{|F\times F|}$. This observaion implies:

Proposition 4.2. *The semigroup* $\mathsf{FRel}(X)$ *endowed with the support map* $\mathsf{supt} : \mathsf{FRel}(X) \to 2^X$ *is* supt*-finitary.*

Finally we check that the supt-semigroup $\mathsf{FRel}(X)$ and some its subsemigroups are supt-perfect.

Proposition 4.3. *For an infinite set X, a subsemigroup $S \subset \mathsf{FRel}(X)$ is* supt*-perfect if for each finite subset $E \subset X$ and each point $x \in X \setminus E$ there is a relation $g \in S(X\backslash E)$ such that $x \notin g(x) \neq \emptyset$.*

Proof. Given a finite subset $F \subset X$ we should check that $S(F) = \mathsf{C}(S(X\backslash F))$. The inclusion $S(F) \subset \mathsf{C}(S(X\backslash F))$ holds because S is a supt-semigroup. To prove that $\mathsf{C}(S(X\backslash F)) \subset S(F)$, take any element $f \in \mathsf{C}(S(X\backslash F))$ and assume that $f \notin S(F)$. Then $\mathsf{supt}(f) \not\subset F$ and we can choose a point $x \in \mathsf{supt}(f) \setminus F$. By our assumption, for the finite set $E = F \cup (\mathsf{supt}(f) \setminus \{x\})$ there is a relation $g \in S(X\backslash E) \subset S(X\backslash F)$ such that $x \notin g(x) \neq \emptyset$. Then $\{x\} \cup g(x) \subset \mathsf{supt}(g)$.

On the other hand, $x \in \mathsf{supt}(f)$ implies $\{x\} \cup f(x) \subset \mathsf{supt}(f)$. Then

$$f(x) \cap g(x) \subset \mathsf{supt}(f) \cap \mathsf{supt}(g) \subset \{x\}$$

and hence $f(g(x)) = g(x)$ and $g(f(x) \setminus \{x\}) = f(x) \setminus \{x\}$.

We claim that $f \circ g \neq g \circ f$. Assuming that $f \circ g = g \circ f$ and taking into account that $x \notin g(x) \neq \emptyset$, we conclude that

$$g(x) = f(g(x)) = g(f(x)) \supset g(f(x) \setminus \{x\}) = f(x) \setminus \{x\}$$

and hence $f(x) \setminus \{x\} \subset (f(x) \setminus \{x\}) \cap g(x) \subset \mathsf{supt}(f) \cap \mathsf{supt}(g) \setminus \{x\} = \emptyset$, which implies that $f(x) \subset \{x\}$.

If $f(x) = \emptyset$, then the set $g(f(x))$ is empty while $f(g(x)) = g(x) \neq \emptyset$. So, $f \circ g \neq g \circ f$.

So, $f(x) \neq \emptyset$ and hence $f(x) = \{x\}$. Then $x \in \mathsf{supt}(f)$ implies that $f^{-1}(x) \neq \{x\}$. Since $x \in f^{-1}(x)$, the set $f^{-1}(x) \neq \{x\}$ is not empty and hence it contains a point $y \in f^{-1}(x) \setminus \{x\}$. It follows that $\{y\} \cup f(y) \subset \mathsf{supt}(f) \subset \{x\} \cup (X \setminus \mathsf{supt}(g))$ and hence $g(y) = \{y\}$ and $x \in f(\{y\}) = f(g(y)) = g(f(y)) = g(f(y) \setminus \{x\}) \cup g(x) = (f(y) \setminus \{x\}) \cup g(x)$, which contradicts $x \notin g(x)$.

This contradiction shows that $g \circ f \neq f \circ g$ and hence $f \notin \mathsf{C}(S(X \setminus F))$ as $g \in S(X \setminus E) \subset S(X \setminus F)$. □

Proposition 4.3 and Corollary 3.5 imply:

Corollary 4.4. *Let X be an infinite set and $S \subset \mathsf{FRel}(X)$ be a subsemigroup such that for each finite subset $E \subset X$ and each point $x \in X \setminus E$ there is a relation $g \in S$ such that $x \notin g(x) \neq \emptyset$ and $\mathsf{supt}(g) \cap E \neq \emptyset$. Then the semigroup S is σ-discrete is its semi-Zariski topology and consequently is σ-discrete in each Hausdorff shift-invariant topology on S.*

This corollary implies the following theorem announced in Abstract.

Theorem 4.5. *Let X be an infinite set X and S be a semigroup such that $\mathsf{FSym}(X) \subset S \subset \mathsf{FRel}(X)$. Then*

1. *S is perfectly supportable;*
2. *S is σ-discrete is its semi-Zariski topology $\mathfrak{Z}_s$;*
3. *S is σ-discrete in each Hausdorff shift-invariant topology on S.*

5. Some Open Problems

The second statement of Theorem 3.2 suggests the following question.

Problem 5.1. *Let (S, supt) be a supt-perfect semigroup and $x \in X$. Is the subset $\{f \in S : x \in \mathsf{supt}(f)\}$ open in the semi-Zariski topology $\mathfrak{Z}_s$? In each Hausdorff semigroup topology on S?*

By Proposition 2.7, each perfectly supportable semigroup has finite double centralizers. We do not know if the converse is also true.

Problem 5.2. *Is each (semi)group with finite double centralizers perfectly supportable?*

The affirmative answer to this problem would imply affirmative answers to the following two problems:

Problem 5.3. *Is each (semi)group with finite double centralizers σ-discrete in its semi-Zariski topology?*

Problem 5.4. *Let S be a semigroup with finite double centralizers and $n \in \mathbb{N}$. Is the set $S_{\leq n} = \{f \in S : |\mathsf{C}(\mathsf{C}(f))| \leq n\}$ closed in the semi-Zariski topology of S? Is it σ-discrete?*

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